

# Light propagation in statistically homogeneous and isotropic universes with general matter content

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**ABSTRACT:** We derive the relationship of the redshift and the angular diameter distance to the average expansion rate for universes which are statistically homogeneous and isotropic and where the distribution evolves slowly, but which have otherwise arbitrary geometry and matter content. The relevant average expansion rate is selected by the observable redshift and the assumed symmetry properties of the spacetime. We show why light deflection and shear remain small. We write down the evolution equations for the average expansion rate and discuss the validity of the dust approximation.

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## 1. Introduction

**A factor of two.** The early universe from at least big bang nucleosynthesis onwards is well described by a model where the geometry is locally spatially homogeneous and isotropic up to linear perturbations, the matter consists of a gas of particles with positive pressure, and the relation between geometry and matter is given by the Einstein equation based on the four-dimensional Einstein-Hilbert action. However, at late times such a model underpredicts the distance to far-away sources and the expansion rate. Compared to the simplest possibility, the spatially flat matter-dominated model, the discrepancy is a factor of about two in both the distance (for a fixed Hubble constant) and the expansion rate (for a fixed energy density or age of the universe). Therefore at least one of the three assumptions –homogeneity and isotropy, standard matter content and standard gravity– is wrong, assuming that light propagation is correctly modeled by null geodesics. No deviations from standard gravity have been observed in local physics, not in the solar system (apart from the Pioneer anomaly and the flyby anomaly, where the possibility of systematics is not ruled out) nor in pulsars [1, 2]. Neither is there any detection of an effect of exotic matter with negative pressure on local physics. The factor of two discrepancy

only appears in observations of distance and expansion rate which involve quantities integrated over large scales<sup>1</sup>. This situation is quite different from that of dark matter, for which there is evidence from various different systems on several scales.

While there is no evidence against standard general relativity or standard matter apart from the increased distance and expansion rate, the universe is known to be locally far from homogeneity and isotropy due to the formation of non-linear structures at late times. It is possible that the breakdown of the homogeneous and isotropic approximation could explain the failure of the prediction of homogeneous and isotropic models with ordinary matter and gravity [4–9]. The effect of inhomogeneity and/or anisotropy on the evolution of the universe was first discussed in detail in [10] under the name “the fitting problem”, and the effect on the average expansion rate is known as backreaction [11–15]. It has been shown with toy models that inhomogeneities can lead to accelerating expansion [14, 16–19], but whether this happens for the distribution of structures present in the real universe is not yet clear. The order of magnitude of the observed change in the expansion rate and the correct timescale of around 10 billion years do emerge from the physics of structure formation in a semi-realistic model [20, 21], but there is no fully realistic calculation yet.

**Light propagation and statistical homogeneity and isotropy.** Most cosmological observations probe quantities related to light propagation, such as the redshift, the angular diameter distance (or equivalently the luminosity distance) and image distortion. In linearly perturbed homogeneous and isotropic Friedmann-Robertson-Walker (FRW) models, the redshift and the distance are to leading order determined by the expansion rate and the spatial curvature. The corrections due to the perturbations are small for typical light rays, and are important only for the cosmic microwave background (CMB), whose redshift anisotropies are very accurately measured, and image distortion, which is zero for the background and remains small when perturbations are included. (At least this is the case when the perturbations are statistically homogeneous and isotropic and the homogeneity scale is small; see [22] for a counterexample with a large spherically symmetric structure.)

The fact that the optical properties of a FRW universe can be expressed in terms of the expansion rate and the spatial curvature is rather obvious, because light propagation is, for small wavelengths, purely geometrical, and these are the only degrees of freedom in the FRW geometry. In a general spacetime, the situation is more involved. Nevertheless, if the distribution of the geometry is statistically homogeneous and isotropic, it could be expected that light propagation over distances longer than the homogeneity scale can to a first approximation be similarly described

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<sup>1</sup>It has been argued that locally repulsive gravity has been observed in the motions of galaxies near the Local Group [3]. This is an interesting possibility, but the present data is not precise enough for such a detection.

with a few quantities related to the overall geometry, regardless of complicated local details [20]. Light propagation in statistically homogeneous and isotropic universes with irrotational dust as the matter content was studied in [23], where it was argued that if the distribution evolves slowly compared to the time it takes for light to cross the homogeneity scale, then the redshift and the angular diameter distance are determined by the expansion rate as a function of redshift and the matter density today. The study [23] had three shortcomings.

First, it was assumed that the variation in the spatial direction of the null geodesics (i.e. light deflection) is small. The magnitude of the null shear was also left undetermined. Observationally, both light deflection and image distortion are known to be small for typical light rays [24], and it should be established that this follows from the symmetry properties of the spacetime. Second, the treatment of matter as irrotational dust is not locally valid [25, 26], because effects such as rotation and velocity dispersion are important for stabilising structures on small scales. The vorticity and non-dust nature of the matter content may be expected to be unimportant for the overall cosmological evolution in the real universe at late times. Nevertheless, such effects should be included to establish under which conditions they can be neglected, and put the dust approximation on better footing. Including matter other than dust is also necessary for treating backreaction in the early universe such as during inflation [12, 27–30] or preheating. Third, the arguments were qualitative and the corrections to the mean behaviour were not determined.

We now remedy the first and second problems. We derive results for light propagation, including deflection and shear, using only assumptions about the symmetry of the spacetime geometry and matter content. We consider general matter content and include rotation. Concentrating on observable quantities related to light propagation, we show how the relevant averaging hypersurface is given by the statistical symmetry of the spacetime. However, our analysis is not more quantitative than [23], and the arguments should be followed up with a more rigorous study.

In section 2 we go through our assumptions, set up the covariant formalism and derive results for the redshift, the deflection, the null shear and the angular diameter distance. In section 3 we derive the evolution equations for the scale factor, which generalise the Buchert equations of the irrotational dust case [11], and consider the validity of the dust approximation. In section 4 we discuss the possible effect of the discreteness of the matter content, the relevance of average quantities and the FRW description, and summarise the situation.

## 2. Light propagation

### 2.1 Spacetime geometry

**Statistical symmetry.** We assume that there exists a foliation of the spacetime

into spatial hypersurfaces of statistical homogeneity and isotropy, which we denote by  $\mathcal{N}$ . The time which is constant on such a hypersurface is denoted  $t$ , and when referring to a particular hypersurface, we use the notation  $\mathcal{N}(t)$ . By this we mean that when we consider any region larger than the homogeneity scale, the average quantities within the region do not depend on its location, orientation or size. In other words, over large scales, there are no preferred locations or directions, and no correlations. Locally the dynamics can be complex, as the assumption of statistical homogeneity and isotropy only concerns average quantities evaluated over large scales. The frame of statistical homogeneity and isotropy may not locally coincide with either the Eckart frame (where there particle number flux is zero) or the Landau-Lifshitz frame (where the energy flux is zero) [31]. However, statistical homogeneity and isotropy does imply that the integrated flux of any quantity through the boundary of a volume larger than the homogeneity scale vanishes.

In this view, the universe consists of identical (up to statistical fluctuations) boxes stacked next to each other. In the real universe, there are correlations even over scales longer than the Hubble scale, due to inflation (or some other process in the early universe) which produces a large region that is exactly homogeneous and isotropic except for linear perturbations. The distribution of the perturbations is statistically homogeneous and isotropic. When the perturbations become non-linear at late times (in typical supersymmetric dark matter models, the first structures form around a redshift of 40–60 [32]), local homogeneity and isotropy are lost, but the distribution of non-linear structures remains statistically homogeneous and isotropic, and the amplitude of correlations is small beyond the homogeneity scale. What one finds as the homogeneity scale depends on the limit that one sets for this amplitude. Based on the fractal dimension of the point set of galaxies, it has been argued that the distribution becomes homogeneous on a scale of around 100 Mpc to an accuracy of about 10% [33]. However, there are still large fluctuations on 100 Mpc scales, and it has been argued that the sample size is not large enough to establish that the distribution is self-averaging, which is a necessary condition for statistical homogeneity [34, 35]. Studies of morphology also suggest that the homogeneity scale could be 300 Mpc or more [36].

Statistical homogeneity and isotropy is formulated in terms of spatial hypersurfaces, but light travels along null geodesics, not in a spacelike direction. Therefore we also need information about the evolution which relates one hypersurface to the next. We assume that the evolution is slow in the sense that the timescale of change in the spatial distribution is much larger than the homogeneity scale. Phrased differently, the variation of the geometry along a null geodesic is rapid compared to the scale over which the mean varies significantly. In the real universe, the timescale for change in the distribution of matter and geometry is the Hubble time  $H^{-1}$ . Today  $H_0^{-1} = 3000h^{-1}\text{Mpc}$  (with  $h$  somewhat below unity [37]), much larger than 100–300 Mpc. In the past, the homogeneity scale was even smaller relative to the Hubble

scale, as structure formation was less advanced.

The combination of statistical homogeneity and isotropy on spatial hypersurfaces and slow evolution from one hypersurface to the next can be heuristically thought of as a distribution that is statistically approximately homogeneous and isotropic in four dimensions when considering scales larger than the homogeneity scale, but smaller than the timescale of change in the distribution. The notion of statistical homogeneity and isotropy in general spacetimes should be made more rigorous, and the role of slow evolution in the arguments we make below on light propagation should be quantified.

**The two frames.** We denote the vector normal to  $\mathcal{N}$  by  $n^\alpha$  and the velocity of the observers by  $u^\alpha$ . Both are normalised to unity,  $n_\alpha n^\alpha = u_\alpha u^\alpha = -1$ . The observer velocity is completely general, it is not assumed to be geodesic or irrotational. For reviews of the covariant approach we use, see [38–42]; for the relation to the ADM formalism [43], see [44]. The tensors which project on the hypersurface orthogonal to  $n^\alpha$  and the rest space orthogonal to  $u^\alpha$  are, respectively,

$$\begin{aligned} h_{\alpha\beta} &\equiv g_{\alpha\beta} + n_\alpha n_\beta \\ h_{\alpha\beta}^{(u)} &\equiv g_{\alpha\beta} + u_\alpha u_\beta, \end{aligned} \quad (2.1)$$

where  $g_{\alpha\beta}$  is the spacetime metric. The restriction of the projection tensor  $h_{\alpha\beta}$  to  $\mathcal{N}$  is the metric on  $\mathcal{N}$ . The spatial derivative of a scalar is defined as  $\hat{\nabla}_\alpha f \equiv h_\alpha^\beta \nabla_\beta f$ , for vectors we have  $\hat{\nabla}_\beta f_\alpha \equiv h_\beta^\delta h_\alpha^\gamma \nabla_\delta f_\gamma$ , and similarly for higher order tensors. The spatially projected traceless part of a tensor is  $f_{\langle\alpha\beta\rangle} \equiv h_{(\alpha}^\gamma h_{\beta)}^\delta f_{\gamma\delta} - \frac{1}{3} h_{\alpha\beta} h^{\gamma\delta} f_{\gamma\delta}$ . The volume element on  $\mathcal{N}$  is  $\epsilon_{\alpha\beta\gamma} \equiv \eta_{\alpha\beta\gamma\delta} n^\delta$ , where  $\eta_{\alpha\beta\gamma\delta}$  is the spacetime volume element. The derivative with respect to the proper time  $s$  of the frame of statistical homogeneity and isotropy is  $n^\alpha \nabla_\alpha$ , and it is denoted by an overdot. Unless  $\dot{n}^\alpha = 0$ , the proper time  $s$  does not coincide with the time  $t$  which is constant on  $\mathcal{N}$ . We can write  $n_\alpha = -\dot{t}^{-1} \partial_\alpha t$ . The derivative with respect to  $t$  is  $m^\alpha \nabla_\alpha$ , with  $m^\alpha = \dot{t}^{-1} n^\alpha$ . We define  $\Gamma \equiv -n_\alpha m^\alpha$ , so  $\Gamma = \dot{t}^{-1} = \partial_t s$  and  $m^\alpha = \Gamma n^\alpha$ . Physically,  $\Gamma$  describes the time dilation due to the non-geodesic motion of the  $n^\alpha$  frame; we have  $\dot{n}_\alpha = \hat{\nabla}_\alpha \ln \Gamma$ . (Note that a  $\Gamma$  which depends only on  $t$  corresponds to a different time coordinate, not different physics.) Only if  $\dot{n}^\alpha = 0$  can we choose  $\Gamma = 1$  and  $s = t$ , which is equivalent to the statement that  $\mathcal{N}$  is a hypersurface of constant proper time. In addition to  $s$  and  $t$ , we also have the proper time of the observers, defined by  $u^\alpha$ .

Because the timescale for the evolution of structures is determined by their proper time, the hypersurface of statistical homogeneity and isotropy could be expected to coincide with the hypersurface of constant proper time of observers co-moving with the structures, as argued in [14, 20, 23]. However, if the matter consists of several components which form structures differently, the situation is not so simple. For example, in the real universe, dark matter and baryons cluster differently (though the differences are not expected to be important on scales larger than the

homogeneity scale). And on small scales, dark matter is multistreaming, so there is more than one proper time associated with the matter flow at a single point. We keep the hypersurface of statistical homogeneity and isotropy arbitrary.

Without loss of generality, we write the observer velocity  $u^\alpha$  in terms of  $n^\alpha$  and a component orthogonal to  $n^\alpha$ ,

$$u^\alpha = \gamma(n^\alpha + v^\alpha) , \quad (2.2)$$

where  $\gamma \equiv -n_\alpha u^\alpha = (1 - v^2)^{-1/2}$ , with  $v^2 \equiv v_\alpha v^\alpha$  and  $v_\alpha n^\alpha = 0$ . Note that  $v^\alpha$  is not the peculiar velocity, either in the perturbation theory sense of a velocity with respect to a fictitious background, or in the physical sense of deviation from a shearfree velocity field [45]. The quantity  $v^\alpha$  measures the deviation of the local observer velocity from the time direction set by the frame of statistical homogeneity and isotropy. Even if  $v^\alpha$  is zero, there can be arbitrarily large spatial variations in the expansion rate. We will see that a large  $v$  implies significant anisotropy in the CMB. We therefore often take  $v$  to be small, and expand to first order in  $v$ . We use  $\simeq$  to indicate equality up to and including terms first order in  $v^\alpha$ . (We do not assume that derivatives of  $v^\alpha$  are small.) Physically, this means that the motion of the observers with respect to the frame of homogeneity and isotropy is non-relativistic.

**Fluid kinematics.** The covariant derivative of  $n^\alpha$  can be decomposed as

$$\nabla_\beta n_\alpha = \frac{1}{3} h_{\alpha\beta} \theta + \sigma_{\alpha\beta} - \dot{n}_\alpha n_\beta , \quad (2.3)$$

where  $\theta \equiv \nabla_\alpha n^\alpha = \hat{\nabla}_\alpha n^\alpha$  is the volume expansion rate and  $\sigma_{\alpha\beta} \equiv \nabla_{\langle\beta} n_{\alpha\rangle} = \hat{\nabla}_{\langle\beta} n_{\alpha\rangle}$  is the shear tensor. The tensor  $\sigma_{\alpha\beta}$  and the acceleration vector  $\dot{n}^\alpha$  are spatial in the sense that they are orthogonal to  $n^\alpha$ ,  $\sigma_{\alpha\beta} n^\beta = 0$ ,  $\dot{n}_\alpha n^\alpha = 0$ . The shear scalar is defined as  $\sigma^2 \equiv \frac{1}{2} \sigma_{\alpha\beta} \sigma^{\alpha\beta}$ . Because  $n^\alpha$  is hypersurface-orthogonal, it follows from Frobenius' theorem that the vorticity  $\omega_{\alpha\beta} \equiv \nabla_{[\beta} n_{\alpha]} + \dot{n}_{[\alpha} n_{\beta]} = \hat{\nabla}_{[\beta} n_{\alpha]}$  is zero [38–40], [46] (page 434).

The covariant derivative of the observer velocity  $u^\alpha$  can be analogously decomposed with respect to itself,

$$\nabla_\beta u_\alpha = \frac{1}{3} h_{\alpha\beta}^{(u)} \theta^{(u)} + \sigma_{\alpha\beta}^{(u)} + \omega_{\alpha\beta}^{(u)} - A_\alpha u_\beta , \quad (2.4)$$

where  $\theta^{(u)} \equiv \nabla_\alpha u^\alpha$ ,  $\sigma_{\alpha\beta}^{(u)} \equiv h_{\alpha\gamma}^{(u)} h_{\beta\delta}^{(u)} \nabla^\delta u^\gamma - \frac{1}{3} \theta^{(u)} h_{\alpha\beta}^{(u)}$ ,  $\omega_{\alpha\beta}^{(u)} \equiv \nabla_{[\beta} u_{\alpha]} + A_{[\alpha} u_{\beta]}$  and  $A^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ .

Given (2.2), the expansion rates in the two frames are related as (see [42] for the expressions for the acceleration, shear and vorticity)

$$\begin{aligned} \theta^{(u)} &= \gamma \theta + \gamma (\hat{\nabla}_\alpha v^\alpha + \dot{n}_\alpha v^\alpha) + \gamma^3 (\dot{v}_\alpha v^\alpha + v^\alpha v^\beta \hat{\nabla}_\alpha v_\beta) \\ &\simeq \theta + \hat{\nabla}_\alpha v^\alpha + \dot{n}_\alpha v^\alpha + \dot{v}_\alpha v^\alpha . \end{aligned} \quad (2.5)$$

**The energy-momentum tensor.** In the geometrical optics approximation, light propagation is kinematical, and independent of the laws which determine the evolution of the geometry. However, we prefer to replace the Einstein tensor with the energy-momentum tensor which describes the matter content, and to do that we assume that the geometry is related to the matter by the Einstein equation,

$$G_{\alpha\beta} = 8\pi G_N T_{\alpha\beta} , \quad (2.6)$$

where  $G_{\alpha\beta}$  is the Einstein tensor,  $G_N$  is Newton's constant, and  $T_{\alpha\beta}$  is the energy-momentum tensor.

Without loss of generality, the energy-momentum tensor can be decomposed with respect to  $n^\alpha$  as

$$T_{\alpha\beta} = \rho^{(n)} n_\alpha n_\beta + p^{(n)} h_{\alpha\beta} + 2q_{(\alpha}^{(n)} n_{\beta)} + \pi_{\alpha\beta}^{(n)} , \quad (2.7)$$

where  $\rho^{(n)} \equiv n^\alpha n^\beta T_{\alpha\beta}$  is the energy density,  $p^{(n)} \equiv \frac{1}{3} h^{\alpha\beta} T_{\alpha\beta}$  is the pressure,  $q_\alpha^{(n)} \equiv -h_\alpha^\beta n^\gamma T_{\beta\gamma}$  is the energy flux and  $\pi_{\alpha\beta}^{(n)} \equiv h_\alpha^\gamma h_\beta^\delta T_{\gamma\delta} - \frac{1}{3} h_{\alpha\beta} h^{\gamma\delta} T_{\gamma\delta} = T_{\langle\alpha\beta\rangle}$  is the anisotropic stress. Both  $q_\alpha^{(n)}$  and  $\pi_{\alpha\beta}^{(n)}$  are spatial in the sense that  $q_\alpha^{(n)} n^\alpha = 0$ ,  $\pi_{\alpha\beta}^{(n)} n^\beta = 0$ . The quantities measured by the observers are given by the decomposition with respect to  $u^\alpha$ ,

$$T_{\alpha\beta} = \rho^{(u)} u_\alpha u_\beta + p^{(u)} h_{\alpha\beta}^{(u)} + 2q_{(\alpha}^{(u)} u_{\beta)} + \pi_{\alpha\beta}^{(u)} , \quad (2.8)$$

where  $\rho^{(u)}, p^{(u)}, q_\alpha^{(u)}$  and  $\pi_{\alpha\beta}^{(u)}$  are defined analogously to the  $n^\alpha$  frame quantities. Locally, dust is defined as matter for which  $p^{(u)}, q_\alpha^{(u)}$  and  $\pi_{\alpha\beta}^{(u)}$  are zero; it then follows from the equations of motion that  $A^\alpha$  is also zero. In the  $u^\alpha$  frame, the non-dust terms have a clear physical interpretation in terms of what the observers measure. Such terms can arise from the properties of matter (it may be that the matter cannot be treated as dust in any frame) and from the fact that an ideal fluid looks non-ideal to a non-comoving observer. We discuss treating the matter approximately as dust in section 3.2.

We could equally take (2.7) and (2.8) as decompositions of the Einstein tensor rather than the energy-momentum tensor. We use assumed symmetry properties of (2.7) such as the absence of preferred directions over large distances, and these could be equally phrased in terms of the geometry expressed in  $G_{\alpha\beta}$ . However,  $T_{\alpha\beta}$  is more transparent because it can be understood in terms of a matter model.

## 2.2 Photon energy and redshift

**The photon momentum.** We want to relate quantities integrated along null geodesics to average quantities which characterise the spatial geometry. We use assumptions about the symmetry properties of the spacetime, so averages are most meaningfully discussed in terms of quantities on  $\mathcal{N}$  and the vector  $n^\alpha$ . In contrast,



the observable redshift and light deflection are defined by the observer velocity  $u^\alpha$ . (The angular diameter distance and the null shear scalar are independent of the velocity field [47].)

In the geometrical optics approximation light travels on null geodesics [48] (page 570), [49] (page 93). We do not consider caustics, which are not expected to be important for typical light rays in cosmology (though see [50]). For treatment of the CMB in the covariant formalism, see [31, 51, 52]. The null geodesic tangent vector is given by the gradient of the phase of the wave, identified with the photon momentum, and denoted by  $k^\alpha$ . It satisfies  $k_\alpha k^\alpha = 0$  and  $k^\alpha \nabla_\alpha k^\beta = 0$ . The redshift plus one is proportional to the energy measured by the observer,  $1 + z \propto E^{(u)}$ , which in turn is

$$E^{(u)} = -u_\alpha k^\alpha . \quad (2.9)$$

The photon momentum can be decomposed into an amplitude and the direction, and the direction can be split into components parallel and orthogonal to  $u^\alpha$ ,

$$k^\alpha = E^{(u)}(u^\alpha + r^\alpha) , \quad (2.10)$$

with  $u_\alpha r^\alpha = 0$ ,  $r_\alpha r^\alpha = 1$ .

Because the vector  $n^\alpha$  is adapted to the symmetry of the spacetime, it is more convenient to calculate quantities in the  $n^\alpha$  frame and then transform to the  $u^\alpha$  frame. The decomposition of  $k^\alpha$  with respect to  $n^\alpha$  reads

$$k^\alpha = E^{(n)}(n^\alpha + e^\alpha) , \quad (2.11)$$

with  $E^{(n)} \equiv -n_\alpha k^\alpha$ ,  $n_\alpha e^\alpha = 0$ ,  $e_\alpha e^\alpha = 1$ . The quantities  $E^{(n)}$  and  $e^\alpha$  do not have a straightforward observational interpretation, unlike  $E^{(u)}$  and  $r^\alpha$ .

The observed energy  $E^{(u)}$  is given in terms of  $E^{(n)}$  by

$$E^{(u)} = \gamma(1 - v_\alpha e^\alpha) E^{(n)} , \quad (2.12)$$

and the observed direction  $r^\alpha$  is related to  $e^\alpha$  by

$$\begin{aligned} r^\alpha &= \frac{1}{\gamma(1 - v_\beta e^\beta)}(n^\alpha + e^\alpha) - \gamma(n^\alpha + v^\alpha) \\ &\simeq (1 + v_\beta e^\beta)e^\alpha + v_\beta e^\beta n^\alpha - v^\alpha . \end{aligned} \quad (2.13)$$

The inverse relation is

$$\begin{aligned} e^\alpha &= \frac{1}{\gamma + v_\beta r^\beta}(u^\alpha + r^\alpha) - \gamma^{-1}u^\alpha + v^\alpha \\ &\simeq (1 - v_\beta r^\beta)r^\alpha - v_\beta r^\beta u^\alpha + v^\alpha . \end{aligned} \quad (2.14)$$

**Statistical homogeneity and isotropy.** We obtain the evolution of  $E^{(n)}$  by operating with the derivative along the null geodesic,  $\frac{d}{d\lambda} \equiv k^\alpha \nabla_\alpha$ . Denoting  $\partial_\eta \equiv (n^\alpha + e^\alpha) \partial_\alpha$  and using (2.3) and (2.11), we have

$$\begin{aligned} E^{(n)} \partial_\eta E^{(n)} &= k^\beta \nabla_\beta E^{(n)} \\ &= -k^\alpha k^\beta \nabla_\beta n_\alpha \\ &= -E^{(n)2} \left( \frac{1}{3} \theta + \dot{n}_\alpha e^\alpha + \sigma_{\alpha\beta} e^\alpha e^\beta \right), \end{aligned} \quad (2.15)$$

which integrates into

$$\begin{aligned} E^{(n)}(\eta) &= E^{(n)}(\eta_0) \exp \left( \int_{\eta_0}^{\eta} d\eta \left[ \frac{1}{3} \theta + \dot{n}_\alpha e^\alpha + \sigma_{\alpha\beta} e^\alpha e^\beta \right] \right) \\ &= E^{(n)}(t_0, \mathbf{x}_0) \exp \left( \int_{t_0}^t dt \Gamma \left[ \frac{1}{3} \theta + \dot{n}_\alpha e^\alpha + \sigma_{\alpha\beta} e^\alpha e^\beta \right] \right) \\ &\approx E^{(n)}(t_0, \mathbf{x}_0) \exp \left( \int_{t_0}^t dt \frac{1}{3} \langle \Gamma \theta \rangle \right), \end{aligned} \quad (2.16)$$

where the integral is along the null geodesic and the subscript 0 refers to the observer's position and time. On the second line we have taken the time  $t$  as the integration variable; the spatial coordinates  $\mathbf{x}$  on  $\mathcal{N}$  are understood as functions of  $t$  on the null geodesic. We have then taken into account that if there are no preferred directions in the geometry of  $\mathcal{N}$  over long distances, and the direction  $e^\alpha$  changes only little or evolves much more slowly than the distribution of the geometry (we discuss this in section 2.3), the dominant contribution is given by the average expansion rate. (We use the symbol  $\approx$  to indicate dropping terms which are suppressed due to statistical homogeneity and isotropy, in contrast to  $\simeq$ , which indicates dropping terms which are small because  $v \ll 1$ .) The argument for this is the following [23]. If  $\Gamma \dot{n}^\alpha$  has no preferred orientation, it points equally in the directions along and opposite to  $e^\alpha$ , so its contribution vanishes. Similarly,  $\Gamma \sigma_{\alpha\beta}$  contributes only via its trace, which is zero. The term  $\frac{1}{3} \Gamma \theta$  then gives the dominant contribution. Under the assumption that the timescale for the evolution of the distribution of the geometry is much larger than the time it takes for light to cross the homogeneity scale, the integral is dominated by the average value of  $\Gamma \theta$ , as the contributions of the variation around the average cancel. (See section 3 for details of the averaging.) In reality, there is some evolution of the quantities along the null geodesic, and the cancellations are not perfect, so the contributions of  $\Gamma \dot{n}_\alpha e^\alpha$ ,  $\Gamma \sigma_{\alpha\beta} e^\alpha e^\beta$  and of the variation of  $\Gamma \theta$  are only suppressed instead of zero.

The observed energy  $E^{(u)}$  is, using (2.12),

$$E^{(u)} \approx E^{(n)}(t_0, \mathbf{x}_0) \gamma (1 - v_\alpha e^\alpha) \exp \left( \int_{t_0}^t dt \frac{1}{3} \langle \Gamma \theta \rangle \right), \quad (2.17)$$

and the redshift  $1 + z = E^{(u)}(\eta)/E^{(u)}(\eta_0)$  is

$$1 + z \approx \frac{\gamma(1 - v_\alpha e^\alpha)}{\gamma_0(1 - v_\alpha e^\alpha)|_0} \exp \left( \int_t^{t_0} dt \frac{1}{3} \langle \Gamma \theta \rangle \right). \quad (2.18)$$

Expressing  $e^\alpha$  in terms of the observed direction  $r^\alpha$  with (2.14), it is transparent that there are large observed anisotropies in the redshift of isotropically distributed sources unless  $v$  is small or constant or there is a conspiracy of cancellations. Conversely, if  $v$  is small, the anisotropy is small, even though the variations in the geometry can be large. In particular, the near-isotropy of the CMB does not imply that the universe would be nearly FRW [23, 53]. Assuming  $v \ll 1$ , the correction due to  $v$  reduces to  $v_\alpha e^\alpha|_0 - v_\alpha e^\alpha \simeq v_\alpha r^\alpha|_0 - v_\alpha r^\alpha$ . The first term is the dipole due to the motion of the observer with respect to the frame of statistical homogeneity and isotropy, and the second, which can have arbitrary angular dependence, is the corresponding term at the source. These are in addition to the usual dipole due to the difference between the velocity of the observer and the source. As long as the difference between  $u^\alpha$  and  $n^\alpha$  is small, the difference in the redshift between the two frames is small, even though the expansion rates  $\theta$  and  $\theta^{(u)}$  can be very different, as the gradient of  $v^\alpha$  can be large even when  $v$  is small.

**The local environment.** When we argue for the cancellation of terms other than  $\langle \Gamma \theta \rangle$  in the integral (2.16) due to symmetry, this only applies to propagation over distances longer than the homogeneity scale, and deviations due to the local environment are not accounted for. For example, in linearly perturbed FRW spacetimes, the shear term  $\sigma_{\alpha\beta} e^\alpha e^\beta$  contains the usual local dipole, which we have neglected. To be consistent in our approximation of concentrating on propagation over long distances and neglecting the effect of the local environments near the source and the observer, we should approximate  $1 - e^\alpha v_\alpha + v_\alpha e^\alpha|_0 \simeq 1$ . In the real universe, this approximation seems to hold well. The velocity difference between the CMB frame and our rest frame is of the order  $10^{-3}$ , and the rest frame of local large-scale structures is also near the CMB frame [54]. The effect of the local environment is likely to be small as long as structures are small compared to the distance the light travels and the observer is not in a special location [20]. This is true for the structures which are known to exist and which are expected in usual models of structure formation, but may not be valid for speculative large spherical structures, often described with the Lemaître-Tolman-Bondi (LTB) model [55, 56].

For the CMB anisotropies, the corrections due to the local environment and the deviations around the mean cannot be neglected. They are important for the low multipoles, as in the Integrated Sachs-Wolfe effect and the Rees-Sciama effect, and could be related [57] to observed violations of statistical isotropy of the CMB [58]. Formalism for the CMB in the case when the geometry is not perturbatively near FRW has been developed in [31].

**The mean redshift and the scale factor.** The redshift characterises a single geodesic (or more accurately, two points and two frames along a single geodesic), so its spatial average is not well defined. However, it is useful to introduce the “mean redshift”  $\bar{z}$  by

$$1 + \bar{z} \equiv \exp \left( \int_t^{t_0} dt \frac{1}{3} \langle \Gamma \theta \rangle \right) . \quad (2.19)$$

The physical interpretation of  $1 + \bar{z}$  is that if we take any two points on  $\mathcal{N}(t)$  and  $\mathcal{N}(t_0)$  which are connected by a null geodesic (or several), the redshift along the null geodesic(s) is  $\bar{z}$  plus small corrections (assuming that the rest frames of the source and the observer are close to the frame of statistical homogeneity and isotropy). From the arguments above and the observational fact that the CMB deviations from isotropy and from the blackbody shape of the spectrum are small [59] we know that temperature differences between different spatial locations are small, and the mean value of the redshift gives the dominant contribution.

We define the scale factor  $a$  as (setting  $a(t_0) = 1$ )

$$a(t) \equiv (1 + \bar{z})^{-1} = \exp \left( - \int_t^{t_0} dt \frac{1}{3} \langle \Gamma \theta \rangle \right) . \quad (2.20)$$

The quantity  $\theta$  gives the change of rate of the local volume element with respect to the proper time  $s$ , so  $\Gamma\theta$  gives the rate of change with respect to  $t$ . Therefore  $a(t)^3$  is proportional to the volume of  $\mathcal{N}$ : the mean redshift is determined by the change of the overall volume of space.

The change of the redshift of a given source with time, called redshift drift [60], has been suggested as a test of the FRW metric and LTB models [61, 62]. Essentially, the change of redshift with time tests the relationship  $1 + z = a(t)^{-1}$  between the redshift and the scale factor, with the scale factor associated with an average expansion rate. In the present case, unlike in LTB models, the relationship between the mean redshift and the average expansion rate is the same as in FRW models. (See section 3.1 for discussion of the average expansion rate.) However, because redshift drift is a small effect, the variations around the mean would have to be considered carefully to make a prediction.

### 2.3 Deflection

**Picard’s proof.** In deriving (2.16) it was assumed that the spatial direction of the null geodesic does not change rapidly along the geodesic. The direction  $e^\alpha$  is the direction of the null geodesic projected on  $\mathcal{N}$ , so it enters into the arguments about cancellation due to symmetry. The direction  $r^\alpha$  in turn describes the apparent position of the source as seen by the observer, so it is an observable, and the change in  $r^\alpha$  is called the deflection. As we do not have information about the ‘original’ position of the source, i.e. the position in the hypothetical situation that the spacetime would

be flat along the photon path, the deflection can only be measured statistically, unless the apparent position changes on the timescale of the observation, as in microlensing.

Let us look at the change of  $e^\alpha$  and  $r^\alpha$  along the null geodesic. As with the redshift, it is simpler to consider  $e^\alpha$  and then relate it to  $r^\alpha$ . It is convenient to choose a non-coordinate basis and introduce tetrads adapted to the 1+3 decomposition [40, 63, 64]. We denote the tetrad basis by  $t^\alpha_A$ , with  $\eta^{AB}t_{\alpha A}t_{\beta B} = g_{\alpha\beta}$ ,  $t^\alpha_A t_{\alpha B} = \eta_{AB}$  as usual, where  $\eta_{AB} = \text{diag}(-1, 1, 1, 1)$ . We use capital Latin letters to denote components in the tetrad basis, e.g.  $e^A \equiv e^\alpha t_\alpha^A$ .

The change of  $e^A$  along the null geodesic is given by

$$\begin{aligned} E^{(n)-1} \frac{de^A}{d\lambda} &= (n^B + e^B) \nabla_B e^A \\ &= n^A \left( \frac{1}{3} \theta + \dot{n}_B e^B + \sigma_{BC} e^B e^C \right) - \dot{n}^A \\ &\quad + e^A (\dot{n}_B e^B + \sigma_{BC} e^B e^C) - \sigma^A_B e^B. \end{aligned} \quad (2.21)$$

where we have on the second line inserted  $e^A = E^{(n)-1} k^A - n^A$  inside the covariant derivative and used (2.3) and (2.15). We specialise the choice of basis by taking  $t_\alpha^0 = n_\alpha$ , so that  $n^A = \delta^{A0}$ ,  $n_A = -\delta_{A0}$ . (In a coordinate basis, this choice is not possible in general [38, 39].) Then  $e^A$  is zero for  $A = 0$ , while for the spatial components (which we denote by small Latin letters from the middle of the alphabet,  $e^i \equiv h^i_A e^A$ ) we obtain, using the definition of the covariant derivative,

$$\begin{aligned} E^{(n)-1} \frac{de^i}{d\lambda} &= (n^B + e^B) \nabla_B e^i \\ &= \partial_\eta e^i + (n^B + e^B) \Gamma^i_{CB} e^C \\ &= \partial_\eta e^i + a^i + \Omega^i_j e^j - a_j e^j e^i - \epsilon^i_{jk} N^j_l e^k e^l, \end{aligned} \quad (2.22)$$

where the connection components  $\Gamma^A_{BC}$  have been expressed in terms of the decomposition (2.3) of  $\nabla_B n_A$  as well as an object  $a_i$ , a symmetric object  $N_{ij}$  and an antisymmetric object  $\Omega_{ij}$ ; see [40, 64] for details<sup>2</sup>. Together with the 9 degrees of freedom in  $\{\theta, \dot{n}^i, \sigma^i_j\}$ , the 12 degrees of freedom in  $\{a^i, N^i_j, \Omega^i_j\}$  completely characterise the spacetime geometry.

Putting (2.21) and (2.22) together, we have a system of ordinary differential equations for the components  $e^i$

$$\begin{aligned} \partial_\eta e^i &= -\dot{n}^i - a^i - (\sigma^i_j + \Omega^i_j) e^j + (\dot{n}_j e^j + a_j e^j + \sigma_{jk} e^j e^k) e^i + \epsilon^i_{jk} N^j_l e^k e^l \\ &\equiv f^i(\eta, e^j). \end{aligned} \quad (2.23)$$

If the geometry is statistically homogeneous and isotropic and its distribution evolves slowly, the change in  $e^i$  remains small due to the lack of preferred directions in

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<sup>2</sup>In the notation of [40, 64],  $\Omega_{ij} \equiv \epsilon_{ijk} \Omega^k$ .

$\{\dot{n}^i, a^i, \sigma^i_j, N^i_j, \Omega^i_j\}$ . This can be expressed more formally as follows. According to Picard's theorem, the system of equations (2.23) has a unique solution given by an iteration (see e.g. [65], page 19). Let us define  $e^i_{(N+1)}(\eta) \equiv e^i_{(0)} + \int_{\eta_0}^{\eta} d\eta' f^i[\eta', e^i_{(N)}(\eta')]$ , with  $e^i_{(0)} \equiv e^i(\eta_0)$ . The solution to (2.23) is given by the  $N \rightarrow \infty$  limit. At first step, we have

$$\begin{aligned} e^i_{(1)} = & e^i_{(0)} - \int_{\eta_0}^{\eta} d\eta' (\dot{n}^i + a^i) - e^j_{(0)} \int_{\eta_0}^{\eta} d\eta' (\sigma^i_j + \Omega^i_j) + e^i_{(0)} e^j_{(0)} \int_{\eta_0}^{\eta} d\eta' (\dot{n}_j + a_j) \\ & + e^i_{(0)} e^j_{(0)} e^k_{(0)} \int_{\eta_0}^{\eta} d\eta' \sigma_{jk} + e^k_{(0)} e^l_{(0)} \int_{\eta_0}^{\eta} d\eta' \epsilon^i_{jk} N^j_l . \end{aligned} \quad (2.24)$$

As with (2.16), we could write  $d\eta = \Gamma dt$ , with the understanding that the spatial coordinates are functions of  $t$  along the null geodesic. If  $\dot{n}^i$  and  $a^i$  have no preferred direction and evolve slowly, their integral vanishes, provided the distance over which the integral is taken is longer than the homogeneity scale. In practice, the cancellation is not perfect, because there is evolution and statistical fluctuations, so the integral is simply suppressed. Similarly, all diagonal components of any tensor should contribute almost equally to the integrals and the contribution from non-diagonal components should be suppressed. Because  $\sigma^i_j, \Omega^i_j$  and  $\epsilon^i_{jk} N^j_l$  are traceless, their contributions are suppressed. One can repeat the argument at every iteration to conclude that the solution  $e^i$  is  $e^i_{(0)}$  plus a small deviation. (For the linear term in (2.23) this is obvious, as it simply gives the  $\eta$ -ordered exponential of the integral of  $\sigma^i_j + \Omega^i_j$ .) The change in the observed position of the source  $r^A$  is then also small as long as  $v$  is small, as we see from (2.13). This qualitative understanding should be made more rigorous, and the amplitude and the distribution of the deflection should be evaluated.

## 2.4 Null shear and angular diameter distance

**The null shear.** The distortion of the size and shape of the source image are described by the null expansion rate  $\tilde{\theta}$  (or equivalently the angular diameter distance  $D_A$ ) and the null shear tensor  $\tilde{\sigma}_{\alpha\beta}$ . To find these quantities, we need to introduce a tensor  $\tilde{h}_{\alpha\beta}$  which projects onto a two-space orthogonal to the null geodesic,  $\tilde{h}_{\alpha\beta} k^\beta = 0$ . Analogously to (2.3) and (2.4), the covariant derivative of  $k^\alpha$  can be decomposed as follows:

$$\begin{aligned} \nabla_\beta k_\alpha &= \tilde{\theta}_{\alpha\beta} \\ &= \frac{1}{2} \tilde{h}_{\alpha\beta} \tilde{\theta} + \tilde{\sigma}_{\alpha\beta} + k_{(\alpha} P_{\beta)} , \end{aligned} \quad (2.25)$$

where the trace  $\tilde{\theta} = \tilde{h}^\alpha_\beta \nabla_\alpha k^\beta = \nabla_\alpha k^\alpha$  is the expansion rate of the area of the null geodesic bundle,  $\tilde{\sigma}_{\alpha\beta} = \tilde{h}^\delta_\alpha \tilde{h}^\gamma_\beta \nabla_\gamma k_\delta - \frac{1}{2} \tilde{h}_{\alpha\beta} \tilde{\theta}$  is the null shear and  $P_\alpha$  is a vector which depends on the choice of  $\tilde{h}_{\alpha\beta}$  and plays no role in what follows. We have  $\tilde{\sigma}_{\alpha\beta} k^\beta = 0$ ,  $P_\alpha k^\alpha = 0$ . The null geodesic vorticity is zero, because  $k^\alpha$  is a gradient. The null

shear scalar is defined as  $\tilde{\sigma}^2 \equiv \frac{1}{2}\tilde{\sigma}_{AB}\tilde{\sigma}^{AB}$ . The area expansion rate  $\tilde{\theta}$  is related to the angular diameter distance by (see e.g. [49, 66])

$$D_A \propto \exp\left(\frac{1}{2} \int d\lambda \tilde{\theta}\right). \quad (2.26)$$

The angular diameter distance  $D_A$  and the shear amplitude  $\tilde{\sigma}_{\alpha\beta}\tilde{\sigma}^{\alpha\beta}$  do not depend on  $\tilde{h}_{\alpha\beta}$ . They also do not depend on the observer velocity, so it is enough to look at the  $n^\alpha$  frame without having to transform to the  $u^\alpha$  frame at the end. It is again convenient to use tetrads instead of sticking to a coordinate basis. We choose  $\tilde{h}_{AB}$  to be orthogonal to both  $n^A$  and  $e^A$ ,

$$\begin{aligned} \tilde{h}_{AB} &= g_{AB} + n_A n_B - e_A e_B \\ &= g_{AB} - E^{(n)-2} k_A k_B + 2E^{(n)-1} k_{(A} n_{B)}, \end{aligned} \quad (2.27)$$

We proceed with  $\tilde{\sigma}_{AB}$  the same way as we did with  $e^A$  in section 2.3. Taking the derivative along the null geodesic and projecting, we obtain (see e.g. [49, 66])

$$\tilde{h}_{(A}{}^C \tilde{h}_{B)}{}^D \frac{d\tilde{\sigma}_{CD}}{d\lambda} = -\tilde{\theta}\tilde{\sigma}_{AB} + C_{AB}, \quad (2.28)$$

where

$$\begin{aligned} C_{AB} &\equiv k^M k^N \tilde{h}_{(A}{}^C \tilde{h}_{B)}{}^D C_{MCND} \\ &= 2E^{(n)2} \tilde{h}_{(A}{}^C \tilde{h}_{B)}{}^D \left( E_{CD} + \frac{1}{2} \tilde{h}_{CD} e^M e^N E_{MN} - \tilde{\epsilon}_{CM} H^M{}_D \right) \end{aligned} \quad (2.29)$$

where  $C_{AB}$  has been expressed in terms of the electric and magnetic components of the Weyl tensor,  $E_{AB} \equiv C_{ACBD} n^C n^D = C_{A0B0}$ ,  $H_{AB} \equiv \frac{1}{2} \epsilon_A{}^{CD} C_{CDBE} n^E = \frac{1}{2} \epsilon_A{}^{CD} C_{CDB0}$ , and  $\tilde{\epsilon}_{AB} \equiv \epsilon_{ABC} e^C$ .

On the other hand, from the definition of the covariant derivative we obtain (see [40, 64] for the decomposition of  $\Gamma^A_{BC}$ )

$$\begin{aligned} E^{(n)-1} \tilde{h}_{(A}{}^C \tilde{h}_{B)}{}^D \frac{d\tilde{\sigma}_{CD}}{d\lambda} &= \partial_\eta \tilde{\sigma}_{AB} - 2\tilde{h}_{(A}{}^C \tilde{h}_{B)}{}^D (n^E + e^E) \Gamma^F_{CE} \tilde{\sigma}_{FD} \\ &= \partial_\eta \tilde{\sigma}_{AB} + (2\Omega_{ij} + 2\tilde{\epsilon}_{k[i} N_{j]}^k + \epsilon_{ijk} N_l^k e^l) \tilde{h}_{(A}{}^i \tilde{\sigma}_{B)}{}^j. \end{aligned} \quad (2.30)$$

Note that in a tetrad basis contracting with  $\tilde{h}_{AB}$  commutes with  $\partial_\eta$ , but not with  $\frac{d}{d\lambda}$ . We denote indices on the space orthogonal to  $n^A$  and  $e^A$  with small Latin letters from the beginning of the alphabet,  $e^a \equiv \tilde{h}^a{}_B e^B$ , and specialise the choice of basis by taking  $t_\alpha{}^3 = e_\alpha$ , so that  $e^A = \delta^{A3}$ ,  $e_A = \delta_{A3}$ . Putting together (2.28) and (2.30), we have

$$\partial_\eta \tilde{\sigma}_{ab} = -E^{(n)-1} \tilde{\theta} \tilde{\sigma}_{ab} + E^{(n)-1} C_{ab} + (2\Omega_{c(a} + 2\tilde{\epsilon}_{d[c} N_{a]}^d + \tilde{\epsilon}_{c(a} N_{3}^3) \tilde{\sigma}_{b)}{}^c. \quad (2.31)$$

Like equation (2.23) for the components of the deflection, (2.31) is an ordinary differential equation for the two independent components of  $\tilde{\sigma}_{ab}$ . However, we cannot straightforwardly apply Picard's theorem. First, (2.31) contains the unknown  $\tilde{\theta}$ . Even if we include the equation (2.32) given below to obtain a closed first order system of equations, Picard's theorem does not apply, because it assumes that the variables remain bounded, whereas the initial condition for the area expansion rate is  $\tilde{\theta}(\eta_0) = -\infty$  at the observer.

Nevertheless, the reasoning about cancellations due the lack of preferred directions still holds, because the solution depends on the source term  $C_{ab}$  only via an integral. This is transparent with the change of variable  $\tilde{\sigma}_{ab} \equiv \tilde{\Sigma}_{ab} + \int d\eta E^{(n)-1} C_{ab}$ . We now argue as before that due to statistical homogeneity and isotropy  $C_{ab}$  contributes dominantly via its trace, which is zero, so  $\tilde{\sigma}_{ab} \approx \tilde{\Sigma}_{ab}$ . This eliminates the source term in (2.31). Given the initial condition  $\tilde{\Sigma}_{ab}(\eta_0) = 0$ , we obtain  $\tilde{\sigma}_{ab} \approx \tilde{\Sigma}_{ab} \approx 0$ . As with the deflection, this argumentation needs to be made more rigorous, and the amplitude of the small shear that is generated should be calculated.

**The angular diameter distance.** Applying the derivative  $\frac{d}{d\lambda}$  to  $\tilde{\theta}$ , we obtain the evolution equation for the area expansion rate (see e.g. [49, 66])

$$\begin{aligned} \frac{d\tilde{\theta}}{d\lambda} &= -G_{AB}k^A k^B - 2\tilde{\sigma}^2 - \frac{1}{2}\tilde{\theta}^2 \\ &= -8\pi G_N T_{AB}k^A k^B - 2\tilde{\sigma}^2 - \frac{1}{2}\tilde{\theta}^2 \\ &= -8\pi G_N (\rho^{(n)} + p^{(n)} - 2q_A^{(n)}e^A + \pi_{AB}^{(n)}e^A e^B)E^{(n)2} - 2\tilde{\sigma}^2 - \frac{1}{2}\tilde{\theta}^2, \end{aligned} \quad (2.32)$$

where we have used the Einstein equation (2.6) and applied the decomposition (2.7) of the energy-momentum tensor. As discussed in section 2.1, we could equally regard (2.7) as the decomposition of the Einstein tensor (or, in the present context, the Ricci tensor, as the trace does not contribute to (2.32)).

Using (2.26), we obtain from (2.32) the evolution equation for the angular diameter distance:

$$\frac{d^2 D_A}{d\lambda^2} = - \left[ 4\pi G_N (\rho^{(n)} + p^{(n)} - 2q_A^{(n)}e^A + \pi_{AB}^{(n)}e^A e^B)E^{(n)2} + \tilde{\sigma}^2 \right] D_A. \quad (2.33)$$

The solution again depends on the source functions only via an integral. This is transparent with the change of variable  $\tilde{\theta} \equiv \tilde{\Theta} - 8\pi G_N \int d\lambda (\rho^{(n)} + p^{(n)} - 2q_A^{(n)}e^A + \pi_{AB}^{(n)}e^A e^B)E^{(n)2}$ . We can write

$$\begin{aligned} &\int d\lambda (\rho^{(n)} + p^{(n)} - 2q_A^{(n)}e^A + \pi_{AB}^{(n)}e^A e^B)E^{(n)2} \\ &\approx E^{(n)}(\eta_0) \int dt a^{-1} \langle \Gamma(\rho^{(n)} + p^{(n)}) \rangle \\ &= E^{(n)}(\eta_0) \int dt a^{-1} \langle \Gamma(\rho^{(u)} + p^{(u)}) + \frac{4}{3}\Gamma F \rangle, \end{aligned} \quad (2.34)$$



where  $F \equiv v^2(\rho^{(u)} + p^{(u)}) + 2\gamma^{-1}q_A^{(u)}v^A + \pi_{AB}^{(u)}v^Av^B \simeq 2q_A^{(u)}v^A$ . On the second line we have taken into account  $d\lambda = E^{(n)-1}d\eta$  and the approximate scaling  $E^{(n)} \approx E^{(n)}(\eta_0)a^{-1}$ . We have also again applied the reasoning that statistical homogeneity and isotropy together with slow evolution implies that the contributions of  $q_A^{(n)}e^A$  and  $\pi_{AB}^{(n)}e^Ae^B$  are suppressed, and that the dominant contribution of  $\rho^{(n)} + p^{(n)}$  comes from the average. Finally, we have written  $\rho^{(n)} + p^{(n)}$  in terms of  $u^\alpha$  frame quantities using (2.1), (2.2), (2.7) and (2.8). Dropping the null shear, the solution  $\tilde{\theta}$  to (2.32) depends only on the quantity (2.34) and  $\lambda = \int d\eta E^{(n)-1} \approx E^{(n)}(\eta_0)^{-1} \int dt a \langle \Gamma \rangle$ , where we have assumed that  $\Gamma$  has a statistically homogeneous and isotropic distribution and varies slowly, so that the integral is dominated by the average value. Because both (2.34) and  $\lambda$  depend approximately only on  $t$  (and  $E^{(n)}(\eta_0)$ ), so does  $\tilde{\theta}$ . Writing (2.32) as an integral equation, dropping subdominant parts which depend on position, taking the time derivative and expressing the equation in terms of the angular diameter distance, we obtain

$$\begin{aligned} H\partial_{\bar{z}} [\langle \Gamma \rangle^{-1} (1 + \bar{z})^2 H\partial_{\bar{z}} \bar{D}_A] &= -4\pi G_N \langle \Gamma(\rho^{(u)} + p^{(u)}) + \frac{4}{3}\Gamma F \rangle \bar{D}_A \\ &\simeq -4\pi G_N \langle \Gamma(\rho^{(u)} + p^{(u)}) + \frac{8}{3}\Gamma q_A^{(u)}v^A \rangle \bar{D}_A, \end{aligned} \quad (2.35)$$

where the notation  $\bar{D}_A(t)$  refers to the dominant part of the angular diameter distance with the corrections to the mean dropped, and we have used the relation  $\partial_t \bar{D}_A = -(1 + \bar{z})H\partial_{\bar{z}} \bar{D}_A$ , with  $H \equiv \partial_t a/a$ . The quantity  $\bar{D}_A$  has a similar physical interpretation as  $\bar{z}$ : if we take any two points on  $\mathcal{N}(t)$  and  $\mathcal{N}(t_0)$  connected by a null geodesic (or several of them), the angular diameter distance along the geodesic(s) is  $\bar{D}_A$  plus small corrections. As noted in [23], while  $\bar{D}_A(\bar{z})$  is well-defined, there does not exist a function  $D_A(z)$  even along a single geodesic, because the redshift is in general not monotonic along the null geodesic<sup>3</sup>.

Apart from  $\Gamma$  and  $F$ , the equation (2.35) for the mean distance in terms of  $\bar{z}$  is the same as in FRW spacetimes. However, the relation between  $\langle \rho^{(u)} + p^{(u)} \rangle$  and  $\bar{z}$  is different than in the FRW case, as we discuss in the next section (see also [23]). As with the redshift and the deflection, it is important to make the arguments about the null shear and the angular diameter distance more rigorous by evaluating the variation around the mean. Observationally, the angular diameter distance is known not to vary much with direction [67].

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<sup>3</sup>In [23] it was incorrectly claimed that when the factor in the parenthesis on the right-hand side of (2.33) is positive,  $D_A$  would be monotonic, because the initial condition (at the observer) for  $\frac{dD_A}{d\lambda}$  is negative. However, because  $\lambda$  decreases along the null geodesic away from the observer, this only implies that  $\frac{dD_A}{d\lambda}$  has at most one zero, so  $D_A$  is separately monotonic on at most two sections of the null geodesic, as is well known from FRW spacetimes. In the present case, the sign of the factor in the parenthesis is not determined, and the number of zeros of  $\frac{dD_A}{d\lambda}$  is not limited.

### 3. The average expansion rate and the scale factor

#### 3.1 The average expansion rate

**Defining the average.** We have started with light propagation, which involves the observed quantities directly, and have been led to consider averages. The average of a scalar  $f$  on  $\mathcal{N}$  is

$$\langle f \rangle(t) \equiv \frac{\int \epsilon f}{\int \epsilon}, \quad (3.1)$$

where  $t$  is constant on  $\mathcal{N}$ . Recall that in general,  $t$  is not a proper time. In particular, it is neither the proper time associated with  $n^\alpha$  nor the proper time measured by the observers. The commutation rule between averaging and taking a derivative with respect to  $t$  is

$$\partial_t \langle f \rangle = \langle \partial_t f \rangle + \langle \Gamma \theta f \rangle - \langle f \rangle \langle \Gamma \theta \rangle \quad (3.2)$$

The scale factor  $a$  was defined in (2.20) to give the mean redshift. From the definition it follows that  $a^3$  is proportional to the volume of  $\mathcal{N}$ . The average expansion rate of interest is

$$\begin{aligned} 3 \frac{\partial_t a}{a} &= \langle \Gamma \theta \rangle \\ &= \langle \Gamma \gamma^{-1} \theta^{(u)} - \hat{\nabla}_\alpha (\Gamma v^\alpha) - \Gamma \gamma^2 (\dot{v}_\alpha v^\alpha + v^\alpha v^\beta \hat{\nabla}_\alpha v_\beta) \rangle \\ &\simeq \langle \Gamma \theta^{(u)} - \hat{\nabla}_\alpha (\Gamma v^\alpha) - \Gamma \dot{v}_\alpha v^\alpha \rangle, \end{aligned} \quad (3.3)$$

where we have used (2.5) and the relation  $\dot{n}_\alpha = \Gamma^{-1} \hat{\nabla}_\alpha \Gamma$ . In the irrotational ideal fluid case, the scale factor was originally defined as the volume of the hypersurface orthogonal to  $u^\alpha$ , or equivalently by using the average of  $\theta^{(u)}$ , with the lapse function included [11, 12]. We start from light propagation, and while  $u^\alpha$  is the relevant velocity field for the redshift, the symmetry of the spacetime selects  $\Gamma \nabla_\alpha n^\alpha = \Gamma \theta$  as the relevant local expansion rate. Locally  $\theta$  can be very different from  $\theta^{(u)}$ , even for small  $v$ , because the derivatives of  $v^\alpha$  can be large. However, using Gauss' theorem [46] (page 433) the total derivative in (3.3) can be converted into a surface integral which describes the flux of  $\Gamma v^\alpha$  through the boundary. If the distribution is statistically homogeneous and isotropic, there should be an equal flux through the surface in both directions, so the integral vanishes (up to statistical fluctuations). Therefore, the difference between the average quantities  $\langle \Gamma \theta \rangle$  and  $\langle \Gamma \theta^{(u)} \rangle$  is suppressed by  $v$ .

#### 3.2 Evolution equations for the scale factor

**The average equations.** We now write down the evolution equations for the scale factor  $a$ , or equivalently for the average expansion rate. These generalise the Buchert equations derived for irrotational dust [11]. In [12] the average equations were written

down in the irrotational ideal fluid case using the ADM formalism, assuming that the averaging hypersurface is orthogonal to the fluid flow<sup>4</sup>. In [69] the average equations were derived (also in the ADM formalism) for an ideal fluid including rotation, taking the expansion rate to be  $h^{\alpha\beta}\nabla_\beta u_\alpha = \theta^{(u)} + n_\alpha \dot{u}^\alpha$ , and keeping the hypersurface of averaging arbitrary. (The formalism was applied to second order perturbation theory in [70], with the hypersurface fixed by the condition  $H_{\alpha\beta} = 0$ . Averaging in second order perturbation theory was also considered in [71], with different hypersurfaces fixed by coordinate conditions.) In [72], the average equations were derived for an ideal fluid in the covariant formalism, with an arbitrary averaging hypersurface. We consider general matter content, an arbitrary hypersurface of averaging and use the covariant formalism.

Combining the Einstein equation (2.6) with the Bianchi and Ricci identities for  $n^\alpha$ , the evolution equations can be conveniently written in terms of the decompositions (2.3) and (2.7) and the electric and magnetic components of the Weyl tensor [40–42]. We are only interested in the three scalar equations

$$\dot{\theta} + \frac{1}{3}\theta^2 = -4\pi G_N(\rho^{(n)} + 3p^{(n)}) - 2\sigma^2 + \dot{n}_\alpha \dot{n}^\alpha + \hat{\nabla}_\alpha \dot{n}^\alpha \quad (3.4)$$

$$\frac{1}{3}\theta^2 = 8\pi G_N \rho^{(n)} - \frac{1}{2}{}^{(3)}R + \sigma^2 \quad (3.5)$$

$$\dot{\rho}^{(n)} + \theta(\rho^{(n)} + p^{(n)}) = -\hat{\nabla}_\alpha q^{(n)\alpha} - 2\dot{n}_\alpha q^{(n)\alpha} - \sigma_{\alpha\beta} \pi^{(n)\alpha\beta} . \quad (3.6)$$

where  ${}^{(3)}R$  is the spatial curvature of  $\mathcal{N}$ . If  $\mathcal{N}$  is a hypersurface of constant proper time, then  $\dot{n}^\alpha = 0$ , and the equations differ from the irrotational dust case only by the pressure term in the Raychaudhuri equation (3.4) and the non-dust terms in the conservation law (3.6). In terms of the Hamiltonian constraint (3.5), the only difference is the different evolution of the energy density. We keep  $\dot{n}^\alpha$  arbitrary. Changing to derivatives with respect to  $t$ , averaging, applying the commutation rule (3.2) and using the relations  $\langle \Gamma\theta \rangle = 3\partial_t a/a$  and  $\dot{n}_\alpha = \Gamma^{-1}\hat{\nabla}_\alpha \Gamma$ , we obtain

$$3\frac{\partial_t^2 a}{a} = -4\pi G_N \langle \rho^{(n)} + 3p^{(n)} \rangle + \langle \dot{n}_\alpha \dot{n}^\alpha \rangle + \langle \hat{\nabla}_\alpha \dot{n}^\alpha \rangle + \mathcal{Q} \\ + \langle \frac{1}{3}(\Gamma^2 - 1)\theta^2 + (1 - \Gamma^{-2})\Gamma\partial_t \theta + \theta\partial_t \Gamma \rangle \quad (3.7)$$

$$3\frac{(\partial_t a)^2}{a^2} = 8\pi G_N \langle \rho^{(n)} \rangle - \frac{1}{2}\langle {}^{(3)}R \rangle - \frac{1}{2}\mathcal{Q} + \frac{1}{3}\langle (\Gamma^2 - 1)\theta^2 \rangle \quad (3.8)$$

$$\partial_t \langle \rho^{(n)} \rangle + 3\frac{\partial_t a}{a} \langle \rho^{(n)} + p^{(n)} \rangle = -\langle \Gamma\theta p^{(n)} \rangle + \langle \Gamma\theta \rangle \langle p^{(n)} \rangle - \langle \Gamma\dot{n}_\alpha q^{(n)\alpha} + \Gamma\sigma_{\alpha\beta} \pi^{(n)\alpha\beta} \rangle \\ - \langle \hat{\nabla}_\alpha (\Gamma q^{(n)\alpha}) \rangle , \quad (3.9)$$

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<sup>4</sup>These average equations, apart from the conservation law of the energy-momentum tensor, were written down in the context of general irrotational matter content in [68]. Note that the perturbative calculation in [68] is incorrect, because the averages of both first order terms and intrinsic second order terms taken in the perturbed spacetime are neglected in comparison with the averages of squares of first order terms. In fact, all these terms are of the same order, and the distinction between them is gauge-dependent [9].

where

$$\mathcal{Q} \equiv \frac{2}{3} (\langle \Gamma \theta^2 \rangle - \langle \Gamma \theta \rangle^2) - 2 \langle \sigma^2 \rangle . \quad (3.10)$$

In [12, 68, 69], the equivalent of the factors of  $\Gamma$  were put inside the averages of the terms which appear on the right-hand side of (3.4) and (3.5), rather than the left-hand side. Inserting (3.4) into the last term of (3.7) and (3.5) into the last term of (3.8) would recover that form of the equations. However, the present convention keeps the  $\hat{\nabla}_\alpha \dot{n}^\alpha$  term and the last term of (3.9) as total derivatives, which we can neglect.

Because the backreaction variable  $\mathcal{Q}$  is a statistical quantity which characterises the distribution of the spatial geometry, it is appropriate to give it in terms of  $\theta$  and  $\sigma$ , which are related to  $n^\alpha$ . However, it seems more appropriate to express the energy-momentum tensor in terms of the decomposition with respect to  $u^\alpha$  from the point of view of estimating the magnitude of the different terms. Using (2.1), (2.2), (2.7) and (2.8), we obtain

$$3 \frac{\partial_t^2 a}{a} = -4\pi G_N \langle \rho^{(u)} + 3p^{(u)} \rangle + \langle \dot{n}_\alpha \dot{n}^\alpha \rangle + \langle \hat{\nabla}_\alpha \dot{n}^\alpha \rangle + \mathcal{Q} \\ + \langle \frac{1}{3} (\Gamma^2 - 1) \theta^2 + (1 - \Gamma^{-2}) \Gamma \partial_t \theta + \theta \partial_t \Gamma \rangle - 8\pi G_N \langle F \rangle \quad (3.11)$$

$$3 \frac{(\partial_t a)^2}{a^2} = 8\pi G_N \langle \rho^{(u)} \rangle - \frac{1}{2} \langle {}^{(3)}R \rangle - \frac{1}{2} \mathcal{Q} + \frac{1}{3} \langle (\Gamma^2 - 1) \theta^2 \rangle + 8\pi G_N \langle F \rangle \quad (3.12)$$

$$\partial_t \langle \rho^{(u)} \rangle + 3 \frac{\partial_t a}{a} \langle \rho^{(u)} + p^{(u)} \rangle = - \langle \Gamma \theta (p^{(u)} + \frac{1}{3} F) \rangle + \langle \Gamma \theta \rangle \langle p^{(u)} + \frac{1}{3} F \rangle \\ - \langle \gamma \Gamma \dot{n}_\alpha q^{(u)\alpha} + \gamma^2 \Gamma (\rho^{(u)} + p^{(u)}) \dot{n}_\alpha v^\alpha + \gamma \Gamma \dot{n}_\alpha q_\beta^{(u)} v^\alpha v^\beta + \Gamma \dot{n}_\alpha \pi^{(u)\alpha\beta} v_\beta \rangle \\ - \langle \Gamma \sigma_{\alpha\beta} \pi^{(u)\alpha\beta} + \gamma^2 \Gamma (\rho^{(u)} + p^{(u)}) \sigma_{\alpha\beta} v^\alpha v^\beta + 2\gamma \Gamma \sigma_{\alpha\beta} q^{(u)\alpha} v^\beta \rangle \\ - \partial_t \langle F \rangle - 4 \frac{\partial_t a}{a} \langle F \rangle - \langle \hat{\nabla}_\alpha (\Gamma q^{(n)\alpha}) \rangle , \quad (3.13)$$

with  $F = v^2(\rho^{(u)} + p^{(u)}) + 2\gamma^{-1} q_\alpha^{(u)} v^\alpha + \pi_{\alpha\beta}^{(u)} v^\alpha v^\beta$  as before. We have not written  $\hat{\nabla}_\alpha (\Gamma q^{(n)\alpha})$  in terms of  $u^\alpha$  frame quantities, because it is suppressed due to statistical homogeneity and isotropy, like  $\hat{\nabla}_\alpha \dot{n}^\alpha$ . Vorticity does not appear in the above equations, because  $n^\alpha$  is hypersurface-orthogonal by construction. Were we to decompose  $\nabla_\beta n_\alpha$  with respect to the  $u^\alpha$  frame, the vorticity of  $u^\alpha$  would (to leading order in  $v$ ) emerge from  $\hat{\nabla}_\alpha \dot{n}^\alpha$  and  ${}^{(3)}R$  (3.4) and (3.5). (For the definition of  ${}^{(3)}R$  for velocity fields which are not hypersurface-orthogonal, see [73].) In particular, the leading order contribution of the  $u^\alpha$  frame vorticity to the average Raychaudhuri equation (3.7) vanishes due to statistical homogeneity and isotropy, because it is contained in the boundary term  $\langle \hat{\nabla}_\alpha \dot{n}^\alpha \rangle$ . (In Newtonian gravity, the vorticity combines with  $\mathcal{Q}$  to give a boundary term, so backreaction vanishes for periodic boundary conditions and for statistical homogeneity and isotropy [74].)

Dropping the boundary terms  $\hat{\nabla}_\alpha(\Gamma q^{(n)\alpha})$  and  $\hat{\nabla}_\alpha \dot{n}^\alpha$  as well as all terms higher than first order in  $v$ , we have

$$3\frac{\partial_t^2 a}{a} \simeq -4\pi G_N \langle \rho^{(u)} + 3p^{(u)} \rangle + \langle \dot{n}_\alpha \dot{n}^\alpha \rangle + \mathcal{Q} \\ + \langle \frac{1}{3}(\Gamma^2 - 1)\theta^2 + (1 - \Gamma^{-2})\Gamma\partial_t\theta + \theta\partial_t\Gamma \rangle - 16\pi G_N \langle q_\alpha^{(u)} v^\alpha \rangle \quad (3.14)$$

$$3\frac{(\partial_t a)^2}{a^2} \simeq 8\pi G_N \langle \rho^{(u)} \rangle - \frac{1}{2} \langle {}^{(3)}R \rangle - \frac{1}{2} \mathcal{Q} + \frac{1}{3} \langle (\Gamma^2 - 1)\theta^2 \rangle + 16\pi G_N \langle q_\alpha^{(u)} v^\alpha \rangle \quad (3.15)$$

$$\partial_t \langle \rho^{(u)} \rangle + 3\frac{\partial_t a}{a} \langle \rho^{(u)} + p^{(u)} \rangle \simeq -\langle \Gamma\theta(p^{(u)} + \frac{2}{3}q_\alpha^{(u)} v^\alpha) \rangle + \langle \Gamma\theta \rangle \langle p^{(u)} + \frac{2}{3}q_\alpha^{(u)} v^\alpha \rangle \\ - \langle \Gamma \dot{n}_\alpha q^{(u)\alpha} + \Gamma(\rho^{(u)} + p^{(u)})\dot{n}_\alpha v^\alpha + \Gamma \dot{n}_\alpha v_\beta \pi^{(u)\alpha\beta} \rangle - \langle \Gamma \sigma_{\alpha\beta} \pi^{(u)\alpha\beta} + 2\Gamma \sigma_{\alpha\beta} q^{(u)\alpha} v^\beta \rangle \\ - 2\partial_t \langle q_\alpha^{(u)} v^\alpha \rangle - 8\frac{\partial_t a}{a} \langle q_\alpha^{(u)} v^\alpha \rangle. \quad (3.16)$$

**The dust approximation.** One reason for deriving the general equations (3.11)–(3.13) is to take into account deviations from the approximation of treating the matter as dust in the late universe. The importance of the different terms depends on the matter model, and cannot be determined from general arguments. However, it is possible to say what would be necessary for the non-dust terms to have a significant effect. For the  $\dot{n}^\alpha$  terms to be important in (3.14)–(3.16),  $\dot{n}_\alpha \dot{n}^\alpha$  would have to be of the order of the square of the expansion rate in a large fraction of space (contrary to what was argued in [75]; see also [76]), or the contraction of the energy flux and  $\dot{n}^\alpha$  would have to be of the order of the product of the average energy density and the average expansion rate. In order for the pressure or the anisotropic stress to be important, they would have to be on average of the same order of magnitude as the average energy density. For the time dilation to be important, the spatially varying part of  $\Gamma$  would have to be of order one in a fraction of space which is of order one. If the matter content is a gas of Standard Model particles plus cold or warm dark matter, and structures evolve from small adiabatic perturbations with a nearly scale-invariant spectrum, it seems unlikely that any of these conditions would be satisfied in the late universe when radiation pressure can be neglected.

Let us assume that the matter is approximately dust in the  $u^\alpha$  frame, i.e. that  $p^{(u)}, q^{(u)\alpha}, \pi_{\alpha\beta}^{(u)}$  and  $A^\alpha$  are small<sup>5</sup>, and the deviation of  $\Gamma$  from unity (and the time derivative of the deviation) is small,  $\Gamma \equiv 1 - \delta\Gamma$ , with  $|\delta\Gamma| \ll 1$ . When we drop all squares of small terms (whether they are non-dust terms or  $v^\alpha$ ), the equations (3.14)–(3.16) simplify to

$$3\frac{\partial_t^2 a}{a} \simeq -4\pi G_N \langle \rho^{(u)} + 3p^{(u)} \rangle + \mathcal{Q} \\ + \langle \dot{v}_\alpha \dot{v}^\alpha + \frac{2}{3}\delta\Gamma\theta^2 + 2\delta\Gamma\partial_t\theta + \theta\partial_t\delta\Gamma \rangle \quad (3.17)$$

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<sup>5</sup>From the equations of motion it follows that  $A^\alpha$  is zero if  $p^{(u)}, q^{(u)\alpha}$  and  $\pi_{\alpha\beta}^{(u)}$  are zero, and  $A^\alpha$  is small if  $p^{(u)}, q^{(u)\alpha}, \pi_{\alpha\beta}^{(u)}$  as well as  $\hat{\nabla}_\alpha p^{(u)}$  and  $\hat{\nabla}^\beta \pi_{\alpha\beta}^{(u)}$  are small.

$$3\frac{(\partial_t a)^2}{a^2} \simeq 8\pi G_N \langle \rho^{(u)} \rangle - \frac{1}{2} \langle {}^{(3)}R \rangle - \frac{1}{2} \mathcal{Q} + \frac{2}{3} \langle \delta \Gamma \theta^2 \rangle \quad (3.18)$$

$$\partial_t \langle \rho^{(u)} \rangle + 3\frac{\partial_t a}{a} \langle \rho^{(u)} \rangle \simeq -\langle \theta p^{(u)} \rangle + \langle \dot{v}_\alpha q^{(u)\alpha} + \rho^{(u)} \dot{v}_\alpha v^\alpha \rangle - \langle \sigma_{\alpha\beta}^{(n)} \pi^{(u)\alpha\beta} \rangle, \quad (3.19)$$

where we have used the fact that  $\dot{n}^\alpha \simeq A^\alpha - \dot{v}^\alpha$  plus corrections of order  $v$ . Equations (3.17)–(3.19) give the leading corrections to the treatment of matter as irrotational dust, compared to the original Buchert equations [11]. In particular, they cover the case when the matter can be locally treated as dust, but has rotation, and  $n^\alpha$  is orthogonal to the hypersurface of constant proper time of observers comoving with the dust. Then  $\dot{v}^\alpha$  is of order  $v$ , and we can choose  $\Gamma = 1$ , so the difference between  $n^\alpha$  and  $u^\alpha$  arises only from vorticity. We see that vorticity alone has (to leading order in  $v$ ) no effect on the averages, because the dominant contribution comes from the total derivative  $\hat{\nabla}_\alpha \dot{n}^\alpha$ . It does change the relation of the scale factor  $a$  to the geometry, because  $a$  will be defined with a different vector field, but not the relation of  $a$  to the redshift.

For irrotational dust, the Raychaudhuri equation (3.4) can be integrated as an inequality before averaging to obtain the bound  $Ht \leq 1$  [46] (page 220), [77, 78]. When rotation or non-dust terms are important, there is no such local inequality. Indeed, having  $\dot{\theta} + \frac{1}{3}\theta^2 > 0$  locally is required in order for collapsing regions to stabilise. From a physical point of view, we would still expect to recover  $Ht \leq 1$  unless there is sustained acceleration in a significant fraction of space, but the conditions for this derived from (3.4) involve combinations of spatial averages and integrals over time, and are not entirely transparent.

Note that in order for the approximation of treating the matter as dust on average to hold, it is only necessary that the contribution of the non-dust terms to the averages is smaller than that of the average energy density. It is not required that the energy-momentum tensor of matter could be locally approximated as dust everywhere. In fact, deviations from the irrotational dust behaviour are necessarily important on small scales. As the local Raychaudhuri equation (3.4) shows, in order to stabilise structures, a large  $\dot{n}^\alpha$  or its gradient is needed. This can correspond to  $u^\alpha$  frame vorticity as with rotating baryonic structures, or the acceleration can be generated by anisotropic stress (or a pressure gradient or energy flux) as in the case of dark matter [26]. For dark matter, the dust approximation is locally invalid in structure formation due to multistreaming [25]. Nevertheless, as long as the volume occupied by regions where such terms are important is small, their contribution to the average is not important. In Newtonian calculations, this is certainly the case [26].

Approximating the matter content as dust on average does not imply viewing the matter as infinitesimal grains. For example, the issue of what the “particles” of the dust fluid are is sometimes raised, and whether one can consistently “renormalise” the scale of the description as larger stable structures form [14, 20]. However, the dust approximation is properly understood as the statement that when considered

on large scales, the energy density dominates over the pressure, the energy flux, the anisotropic stress, and their gradients.

It has been argued that because of gradients of spatial curvature, clocks in different regions run at different rates, and that this effect is important for cosmology but neglected in the dust approximation [79]. Any such effects are accounted for in the present analysis, to the extent they are part of general relativity, and not outside of the geometrical optics approximation. If the  $n^\alpha$  frame is non-geodesic, different points on the hypersurface of statistical homogeneity and isotropy indeed have different values of the  $n^\alpha$  frame proper time (though this cannot be understood as being due to spatial curvature gradients), which in turn is close to the proper time measured by the observers if  $v$  is small. For this time dilation to be significant, the acceleration  $\dot{n}^\alpha$  would have to be of the order of the average expansion rate in a significant fraction of space. This in turn requires that either the motion of the observers is very non-geodesic (large  $A^\alpha$ ) or the acceleration between the two frames is significant (large  $\dot{v}^\alpha$ ); the latter possibility however has to contend with the fact that the velocity between the frames cannot become large, as this would violate the small anisotropy observed in the CMB. In order to generate such large accelerations, the non-dust terms in the energy-momentum tensor would have to be significant in a large fraction of space. This would likely have important cosmological effects apart from the time dilation. Note that it follows from statistical homogeneity and isotropy that the spatial difference in the CMB temperature between different regions is small, in contrast to the arguments made in [79]. This issue can be observationally probed with the blackbody shape of the CMB spectrum [59, 80].

## 4. Discussion

### 4.1 Modelling issues

**Discreteness.** While we have kept the energy-momentum tensor generic, the arguments about light propagation in section 2 contain the implicit assumption that matter is so finely distributed in space that it can be treated as a continuous distribution which light rays sample. The redshift, the angular diameter distance and other quantities related to light propagation depend on the spacetime geometry only via an integral along the null geodesic. If the matter consists of discrete clumps whose size and number density is so small that a typical light ray will never encounter matter, the energy-momentum tensor along the light ray is zero, regardless of the average energy density (or pressure or other components). For example, the integrand in (2.34) vanishes identically, and our arguments about cancellation between regions of low and high density do not apply.

The approximation of discrete matter as a continuous fluid has been studied from first principles for the dynamics of matter in Newtonian gravity [81]. However, the

effect on light propagation has been looked at mostly from the perspective of adding perturbations to a FRW metric. With a statistically homogeneous and isotropic distribution and small structures, it is then not surprising that the deviation of the expansion rate from the FRW case is small [20]. One exception is [82], where the effect of discreteness light propagation was considered in a lattice model without any FRW approximation.

If the light travels in vacuum, and we assume statistical homogeneity and isotropy, the mean angular diameter distance is given by (2.35) with zero on the right-hand side. The equation can be integrated to yield

$$\bar{D}_A = \int_0^{\bar{z}} dz \frac{\bar{\Gamma}(z)}{(1+z)^2 \bar{H}(z)} , \quad (4.1)$$

where  $\bar{H}$  and  $\bar{\Gamma}$  are the mean expansion rate and the mean time dilation along the null geodesic, which in general do not coincide with the spatial averages. Because the null geodesic samples only vacuum, the expansion rate along the null geodesic is larger than the average expansion rate (assuming that the matter satisfies the strong energy condition). The equations (3.4) and (3.5) show that if the acceleration  $\dot{n}^\alpha$  and the shear  $\sigma_{\alpha\beta}$  could be neglected, the expansion rate sampled by the light ray would be the same as in an empty FRW universe. If the time dilation could be neglected, the angular diameter distance would then correspond to the 'coasting universe' (or Minkowski space, for non-expanding regions). However, it is probably not reasonable to neglect the shear  $\sigma_{\alpha\beta}$ . For example, the existence of both expanding and non-expanding regions means that there is a gradient in the expansion rate, which implies non-zero shear. Evaluating the expansion rate along a null geodesic, and the angular diameter distance, is thus reduced to the question of realistically modelling the shear scalar (and the acceleration  $\dot{n}^\alpha$ ) along the geodesic.

Discussing light propagation in terms of null geodesics assumes the validity of the geometrical optics approximation. Geometrical optics is in turn based on modelling light as local plane waves, which requires the wavelength of the light to be much smaller than both the curvature scale and the scale over which the amplitude, wavelength and polarisation of the light change significantly. If the fraction of the volume occupied by matter is so small that light rays never come close to the matter particles, this is satisfied. However, if the light passes through small discrete regions where the energy-momentum tensor is non-zero, the situation is very different from the geometrical optics limit. (We are here concerned only with gravitational interactions, and are not taking into account gauge interactions between photons and matter.) In evaluating the validity of the continuous fluid approximation, the extension of the wave-packet of the matter particles should therefore be taken into account. The treatment of the photon waves should also be more detailed, instead of simply treating their transverse width as zero (as in the null geodesic picture) or



infinite (as implicit in the plane wave approximation). The effect of discreteness on light propagation is not obvious, and should be studied in a realistic model.

**The relevance of averages.** The equations (3.7)–(3.9) or (3.11)–(3.13) generalise the average equations for the irrotational dust case derived by Buchert [11] to arbitrary matter content and rotation, and an arbitrary averaging hypersurface. The scale factor has been defined to give the mean redshift in the case that the difference between the observer frame and the frame of statistical homogeneity and isotropy is small, and we have assumed statistical homogeneity and isotropy and slow evolution, which are required to have a meaningful notion of mean redshift. The equations (3.7)–(3.9) or (3.11)–(3.13) are of course valid without any symmetry assumptions. However, the quantity  $a^3$  is of limited use in interpreting observations unless the space is statistically homogeneous and isotropic, so that the change in the total volume of the spatial hypersurface is the dominant effect for light propagation.

The system of equations (3.7)–(3.9) or (3.11)–(3.13) is derived from the scalar part of the full Einstein equation, which is not closed, because it is coupled to the vector and tensor parts. As the sum of two tensors at different spacetime points is not a tensor, the average of a tensor (or vector) in curved spacetime is not well-defined, so it is sometimes said that the rest of the evolution equations cannot be averaged. However, we can write the evolution equations in terms of components, and average these. The problem is not lack of covariance<sup>6</sup>, but the fact that products of variables become independent correlation terms, so the number of unknowns increases when taking the average, and the set of average equations does not close.

Methods for covariantly averaging tensors on curved spacetime have been suggested, including the macroscopic gravity formalism [84, 85], the Ricci flow [86], a statistical averaging formalism [87], a procedure which relies on a specific choice of tetrads [88], and the proposal of [89] which is more a way to rewrite the tensors than average them. However, the relevant issue is not the mathematical definition of averages in some covariant formalism which is an extension of general relativity or in a statistical ensemble of spacetimes. Rather, we want to determine the impact of structures in the spacetime which actually describes the universe we observe, with the dynamics determined by the Einstein equation and the local equations for light propagation. Averages are useful insofar as they provide an approximate description of observed quantities in this complex system. The relevant averaging procedure, and the hypersurface of averaging, emerges from considering observations, and cannot be determined on abstract mathematical grounds.

It bears emphasising that taking the average on a different hypersurface would correspond to considering a different velocity field, and this is a physical choice. This

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<sup>6</sup>One can even average the vector and tensor part of the equations covariantly by first projecting with a vector field such as  $v^\alpha$  or  $\hat{\nabla}_\alpha \theta$ . The Einstein equation can be expressed in full generality in scalar form by using a projection [83].

issue is separate from the question of gauge-invariance, i.e. dependence of unphysical quantities on the chosen coordinate system. The Buchert equations were originally derived using the ADM formalism [11], where the distinction between choice of velocity field and choice of coordinates is not entirely transparent. However, the problem can be considered completely covariantly, without introducing coordinates [20, 23, 42]. The averages depend on the choice of the averaging hypersurface [29, 90], but not on the coordinate system [9]<sup>7</sup>. The relevant velocity field is singled out as that of the observer by the redshift, and the relevant averaging hypersurface is given by the symmetry properties of the spacetime.

**Deviation from the FRW universe.** If backreaction is important for the average expansion rate, i.e. if  $\mathcal{Q}$  contributes significantly to (3.7) and (3.11), there is no “FRW background” that would emerge on large scales [14, 20]. (Note that  $\mathcal{Q}$  being small does not guarantee that the spacetime can be described by the FRW metric.) The FRW metric describes a universe that is exactly homogeneous and isotropic, not a universe where there are large non-linearities with a statistically homogeneous and isotropic distribution.

While the deviation of the average expansion rate from the FRW equations could be attributed to an effective matter component in a FRW universe, this is not the case for other observables. For the shear scalar, this is obvious, because it is zero in FRW models, but generally positive. As a less trivial example, the spatial curvature in the FRW case is fixed to be proportional to  $a^{-2}$ , while the average spatial curvature in an inhomogeneous and/or anisotropic space can evolve non-trivially [91]. In fact, if the matter can be treated as dust, the effect of backreaction is encoded in the non-trivial evolution of the spatial curvature [11, 14, 78]. (The difference between backreaction in general relativity and Newtonian gravity can also be understood in terms of the spatial curvature [20].) For this reason, calling the effect of backreaction a change of background as in [70, 92, 93] is misleading. A FRW model which reproduces the average expansion rate of a clumpy model is better called a “fitting model” or something similar. The metric associated with it cannot be used to calculate quantities other than the one specifically fitted for, and usual perturbation theory around it does not make sense.

In particular, if backreaction is important, the relation between the expansion rate and the angular diameter distance given by (2.35) (assuming that discreteness is not important) is different from the FRW case [23]<sup>8</sup>. Even though either the change of the expansion rate or the change of the distance due to backreaction can be

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<sup>7</sup>In [71], choice of the averaging hypersurface and the coordinate system was conflated. This mixes up defining a quantity of interest using physical criteria and using different coordinates to describe the same physics.

<sup>8</sup>In contrast, the relation  $D_L = (1+z)^2 D_A$  for the luminosity distance  $D_L$  is universal [39], [49] (page 111), [94].

reproduced in a FRW model by introducing extra sources of matter or changing the Einstein equation, it is not possible to do both at the same time, since FRW models cannot mimic the correlation (2.35). In a clumpy space, if the dust approximation holds, the distance is uniquely determined by the function  $H(\bar{z})$  and the matter density today [23]. (Note that fitting the distance observations may not necessarily require accelerating expansion, because the relation between  $H$  and  $D_A$  is different from the FRW case.) Analogously, in a FRW universe with general matter content, the distance is determined by  $H(z)$  and the spatial curvature today [95]. In LTB models, the relationship is different from either of these cases [56].

This prediction for both FRW models and backreaction can be tested with independent observations of distance and the expansion rate [96]. The deviation of the backreaction case from the FRW consistency condition is related to the difference of the average expansion rate from the FRW case with vacuum energy and dust [23]. There are relatively good constraints on the distance scale as a function of redshift from type Ia supernova observations [97], but measurements of the expansion rate using the ages of passively evolving galaxies are less precise [98]. The expansion rate at different redshifts also enters into the radial mode of the baryon acoustic oscillations, and a measurement was reported in [99] (see also [35, 100]). With better observations of the expansion rate, it will be possible to more tightly test the statement that the universe is described by a FRW metric, independent of the possible existence of exotic matter or modified gravity [95, 101]<sup>9</sup>. Similarly, backreaction can be tested without having a prediction for the average expansion rate. In the case of backreaction, it helps that there are independent observational constraints on the matter density today, while the only way to determine the spatial curvature of a FRW universe is to make independent measurements of the expansion rate and distance, and use the FRW relation between them. (In particular, the CMB has no model-independent sensitivity to the spatial curvature.)

## 4.2 Conclusions

**Summary.** In [23] it was argued that light propagation can be approximately described in terms of the overall geometry (meaning the average expansion rate and other average scalar quantities) in statistically homogeneous and isotropic dust universes where the distribution evolves slowly compared to the time it takes for light to cross the homogeneity scale. The calculation was incomplete because it was simply assumed that the light deflection is small, and there was no result for the amplitude of the image shear. It was also assumed that the matter is dust and irrotational, while it is known that such a description does not locally hold everywhere.

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<sup>9</sup>Note that this is not a test of the Copernican principle. The statement that our position in the universe is not untypical is different from the statement that the metric is FRW. In fact, the Copernican principle says nothing about the metric.

We have now considered general matter content, with a general observer velocity and a general hypersurface of statistical homogeneity and isotropy, with a slowly evolving distribution. From these assumptions about the spacetime symmetry, we find that the propagation of typical light rays over distances longer than the homogeneity scale can to leading order be treated in terms of a few average quantities, as in the irrotational dust case. The redshift is given by the average volume expansion rate of the hypersurface of statistical homogeneity and isotropy, assuming that the velocity difference between the frame of statistical homogeneity and isotropy and the observer frame is non-relativistic. The relevant averaging hypersurface is selected by the symmetry of the situation as the one of statistical homogeneity and isotropy, while the relevant velocity is that of the observers, because it gives the observed redshift. The angular diameter distance is to leading order determined by the averages of the expansion rate, time dilation, energy density and pressure. The light deflection and the image shear are small.

We have also written down the generalisation of the Buchert equations [11] for the evolution of the average expansion rate to general matter content and averaging hypersurface. Provided that the difference between frame of statistical homogeneity and isotropy and the observer frame is small, and that pressure, energy flux and anisotropic stress are not significant in a large fraction of space, we recover the Buchert equations for dust.

**Outlook.** If backreaction has a large effect on the average expansion rate, the relation between the expansion rate and the angular diameter distance is different from the FRW case, and the difference grows with the deviation of the expansion rate from that of the FRW model with dust and vacuum energy. This is a distinct prediction of backreaction, which cannot be mimicked by any FRW model [23].

Our arguments are qualitative, and should be followed up by a more rigorous quantitative study. In particular, evaluating the deviations from the mean is necessary to study lensing and the low multipoles of the CMB. The equations we have written in the covariant formalism contain all general relativistic effects and are exact, except for the geometrical optics approximation. The study of light propagation in a general spacetime is reduced to the system of coupled ordinary differential equations (2.15), (2.23), (2.31) and (2.32). To obtain a solution, we do not have to know the global geometry, it is only necessary to specify the distribution along the null geodesic. This approach was used in the perturbative case in [102]. This formulation is well-suited to slowly evolving statistically homogeneous and isotropic universes, where the solution is expected to depend only on the statistical properties of the distribution. In particular, because in the real universe the distribution originates from small almost Gaussian fluctuations, its statistics are even at late times determined by the initial power spectrum, processed by gravity. The effect of the discreteness of the matter content of the universe on light propagation should be clarified. Finally,

even though the relation between the expansion rate and the distance is already a prediction which can be checked, it remains of central importance to derive the average expansion rate from the statistics of structure formation [20] in a reliable manner to allow easier and more comprehensive comparison with observations.

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